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Quasiperiodic plane tilings based on stepped surfaces

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Static and dynamic characteristics of layerwise growth in two-dimensional quasiperiodic Ito–Ohtsuki tilings are studied. These tilings are the projections of three-dimensional stepped surfaces. It is proved that these tilings have hexagonal self-similar growth with bounded radius of neighborhood. A formula is given for the averaged coordination number. Deviations of coordination numbers from its average are quasiperiodic. Ito–Ohtsuki tiling can be decomposed into one-dimensional sector layers. These sector layers are one-dimensional quasiperiodic tilings with properties like Ito–Ohtsuki tilings.

1. Introduction

The 'cut-and-project' method is a well known approach for construction of quasiperiodic tilings. Main examples are the de Bruijn construction for Penrose tilings (de Bruijn, 1981) and the concept of 'model set' (Meyer, 1972; Moody, 2000). Recently, Ito & Ohtsuki (1994) proposed a new construction of quasiperiodic tilings based on the periodic tiling of three-dimensional space.

Ito–Ohtsuki tilings are a family of plane tilings. Every tiling from this family consists of three types of parallelograms. Destainville *et al.* (2000) and Vidal & Mosseri (2000) give some other examples of parallelogram plane tilings. A well known 'wierigna roof' tiling is also an example of tilings of such a type.

In recent years, some dynamic characteristics of periodic and non-periodic tilings have been studied. These investigations are based on the layerwise growth model (Rau *et al.*, 2002). Periodic (Zhuravlev, 2002) and 1-periodic (Shutov, 2003) structures have the polygonal growth form (a graph or tiling is called 1-periodic if its lattice of translations is one-dimensional). If we consider non-periodic structures, we can obtain elliptic arcs in growth form (Zhuravlev *et al.*, 2002). Zhuravlev & Maleev (2007) studied layerwise growth of quasiperiodic Rauzy tiling (Rauzy, 1982). This tiling also has the polygonal growth form.

Another important dynamic characteristic of the tiling is its coordination sequence (Grosse-Kunstleve *et al.*, 1996). The coordination sequence is strictly connected with the layerwise growth model. It is possible to give exact formulas for coordination numbers in a periodic case (Shutov, 2005). For quasiperiodic tilings, only asymptotic formulas can be obtained (Baake & Grimm, 2003). Zhuravlev & Maleev (2008) studied more thoroughly the properties of the coordination sequence for a Rauzy tiling. One of these characteristics is the deviation of a coordination number from its average. Such deviations are quasiperiodic. Quasiperiods are determined by the continued fraction expansions of some algebraic Pisout numbers.

The study of Rauzy tiling is based on its parameterization from the paper of Zhuravlev & Maleev (2007). This parameterization is based on the branching dynamic system on a one-dimensional torus.

The aim of this paper is to prove the analogs of described results for Ito–Ohtsuki tilings. Firstly, we obtain a new parameterization of these tilings. Further, we study layerwise growth and coordination sequence of Ito–Ohtsuki tilings. We use some ideas and methods from number theory and the theory of dynamic systems, and computer modeling.

2. Stepped surfaces and quasiperiodic tilings

Ito & Ohtsuki (1994) proposed the following construction. Consider the normal tiling T^3 of three-dimensional space by unit cubes. Vertices of all cubes form an integer lattice with the basis $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$. Consider four numbers a, b, c, h such that $0 < a < b < c$. Suppose that a, b and c are linearly independent over the field of rational numbers. For every cube from T^3, we consider its vertex with the following property: the vector from the vertex to the center of this cube is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. This vertex is called a labeled point of the cube. By definition, coordinates of a cube are coordinates of its labeled point. For example, the cube $\lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \lambda_3\mathbf{e}_3$: any $\lambda_i \in [0, 1)$ has coordinates $(0, 0, 0)$. Let P be a plane $ax + by + cz + h = 0$. Let S be a set of cubes from the half-space $ax + by + cz + h \leq 0$. The boundary B of the set S consists of the faces of unit cubes. The set B is called a stepped surface.

It can be proved that the face with a labeled point (p, q, r) lies on the stepped surface iff $0 < ap + bq + cr + h \leq a + b + c$. Consider the plane P_0: $x + y + z = 0$. Let $\boldsymbol{\pi}$ be an orthogonal projection from three-dimensional space to P_0. Let $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\pi}_3$ be the images of the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ under this projection. Then $\boldsymbol{\pi}_1 + \boldsymbol{\pi}_2 + \boldsymbol{\pi}_3 = 0$ and two vectors $\boldsymbol{\pi}_1$, $\boldsymbol{\pi}_2$

generate a hexagonal lattice in the plane. Projection gives one-to-one correspondence between vertices of the stepped surface B and points of the hexagonal lattice. This correspondence is given by the following formulas:

$$(p, q, r) \in B \rightarrow (p - r)\pi_1 + (q - r)\pi_2 \in \Gamma,$$
$$m\pi_1 + n\pi_2 \in \Gamma \rightarrow (m, n, 0) + \left(1 - \left\lceil \frac{am + bn + h}{a + b + c} \right\rceil\right)(1, 1, 1) \in B.$$

Here $\lceil x \rceil$ is the minimal integer that is greater than or equal to x. This implies that different points of the stepped surface B are projected into different points of the plane P_0. It means that the projection $\pi(B)$ is the tiling $\mathrm{Til}(a, b, c, h)$ of the plane P_0.

From the linear independence of the numbers a, b, c over the field of rational numbers follows the quasiperiodicity of the tiling $\mathrm{Til}(a, b, c, h)$.

Remark 1. If the numbers a, b, c are not linearly independent, then the tiling $\mathrm{Til}(a, b, c, h)$ has a one- or two-dimensional lattice of its translation symmetries.

Remark 2. Pytheas Fogg (2002) gives an alternative construction of the tilings $\mathrm{Til}(a, b, c, h)$ based on an infinite sequence of substitutions.

Remark 3. This construction was also given by Arnoux *et al.* (2002).

In Fig. 1, we represent a patch of the quasiperiodic tiling $\mathrm{Til}(a, b, c, h)$ with $a = 1, b = \sqrt{2}, c = \sqrt{3}, h = 0.1$. Further, we will give all numeric data for this example.

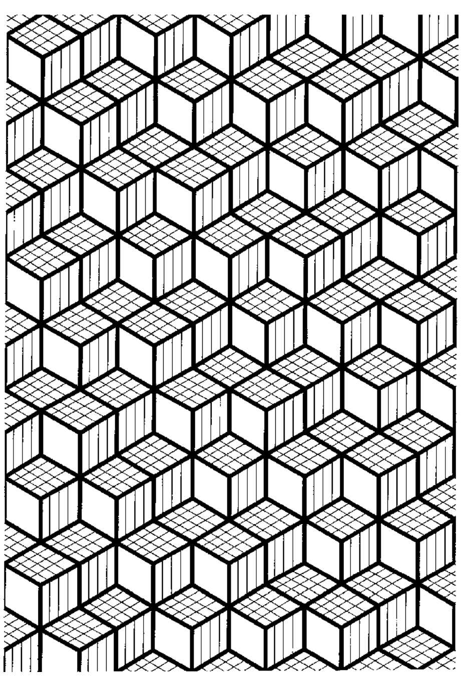

Figure 1
A patch of a two-dimensional quasiperiodic tiling $\mathrm{Til}(1, \sqrt{2}, \sqrt{3}, 0.1)$.

3. Weak parameterization

The tiling $\mathrm{Til}(a, b, c, h)$ is quasiperiodic. Its tiles are rhombs of three orientations with labeled points (see Fig. 2). Mathematically, these rhombs are

$$E_1 = \{\lambda\boldsymbol{\pi}_2 + \mu\boldsymbol{\pi}_3\},\ E_2 = \{-\lambda\boldsymbol{\pi}_1 + \mu\boldsymbol{\pi}_3\},\ E_3 = \{-\lambda\boldsymbol{\pi}_1 - \mu\boldsymbol{\pi}_2\}, \quad \text{where } \lambda, \mu \in [0, 1).$$

We say that the rhombs of types 1, 2 and 3 are red, green and blue, respectively. Weak parameterization gives a correspondence between coordinates of labeled points $(m, n) = m\boldsymbol{\pi}_1 + n\boldsymbol{\pi}_2$ from the hexagonal lattice and types of its rhombs. The set of parameters is a half-interval $I_0 = (0, a + b + c]$. Consider the map U:

$$U(m, n) \rightarrow \begin{cases} (am + bn + h) \bmod (a + b + c), \\ \quad \text{if } (am + bn + h) \bmod (a + b + c) \neq 0 \\ a + b + c, \text{otherwise.} \end{cases}$$

Using this map, we obtain the parameter from I_0 for every rhomb from $\mathrm{Til}(a, b, c, h)$. Here $x \bmod y = x - y[x/y]$ and $[x]$ is the integer part of x.

Now we can describe the sets of parameters which correspond to rhombs of every type. The set of parameters I_0 decomposes into three half-intervals $I_1 = (0, a]$, $I_2 = (a, a + b]$, $I_3 = (a + b, a + b + c]$. The rhomb with labeled point $(m, n) = m\boldsymbol{\pi}_1 + n\boldsymbol{\pi}_2$ is red if $U(m, n) \in I_1$, green if $U(m, n) \in I_2$ and blue if $U(m, n) \in I_3$.

The weak parameterization of Ito–Ohtsuki tiling can be found in the paper of Arnoux *et al.* (2002). This parameterization produces a very convenient computer algorithm for construction of $\mathrm{Til}(a, b, c, h)$.

4. Strong parameterization

The rhombs from the tiling $\mathrm{Til}(a, b, c, h)$ are called neighboring if they have a common edge. Now our aim is to determine all neighboring rhombs for each rhomb. Recall that we have correspondence between a rhomb and its parameters from I_0. So we must determine the parameters of neighboring rhombs from the parameter of one rhomb. Weak parameterization cannot solve this problem.

For strong parameterization, we glue left- and right-hand parts of the half-interval I_0. Now we can consider I_0 as a one-dimensional torus. Consider the following operation:

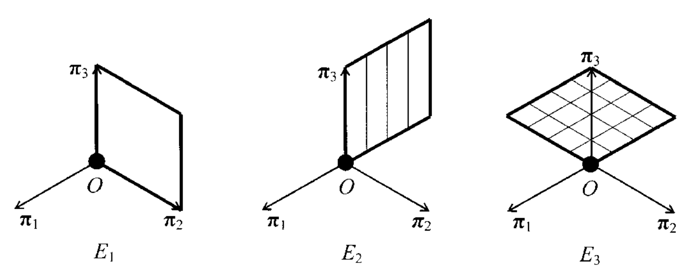

Figure 2
Three types of rhombs from the tiling $\mathrm{Til}(a, b, c, h)$ and its labeled points.

Table 1
Strong parameterization of the tiling Til(a, b, c, h).

$c < a + b$		$c > a + b$	
Half-interval	Local numbers	Half-interval	Local numbers
$(0, a]$	$\pm a, b, -c$	$(0, a]$	$\pm a, b, -c$
$(a, b]$	$\pm a, b, c$	$(a, b]$	$\pm a, b, c$
$(b, a + b]$	$-a, \pm b, c$	$(b, a + b]$	$-a, \pm b, c$
$(a + b, a + c]$	$a, \pm b, c$	$(a + b, 2a + b]$	$a, \pm b, c$
$(a + c, 2a + b]$	$a, -b, \pm c$	$(2a + b, a + c]$	$\pm a, \pm b$
$(2a + b, a + b + c]$	$\pm a, -b, -c$	$(a + c, a + b + c]$	$\pm a, -b, -c$

$$x \oplus y = \begin{cases} (x + y) \bmod (a + b + c), \\ \quad \text{if } (x + y) \bmod (a + b + c) \neq 0 \\ a + b + c, \text{ otherwise.} \end{cases}$$

Let us define local numbers $\{\omega_i\}_{i=1}^4$ of the point x from the parameter set I_0 by the following property: $\{x \oplus \omega_i\}$ is the set of parameters of the rhombs neighboring the rhomb with the parameter x. Strong parameterization gives the correspondence between the points from I_0 and their local numbers. Strong parameterization depends on the sign of $c - (a + b)$. In Table 1, we give half-intervals and corresponding local numbers which determine the strong parameterization.

5. Layerwise growth

In the paper of Zhuravlev *et al.* (2002), the authors proposed a layerwise growth model for tilings and packings as a simple geometrical model for crystal growth. This model means the following. Let Til be a tiling. Consider some finite set of tiles from the Til. This set is called the seed. Tiles neighboring the tiles from the seed form the first coordination encirclement. Similarly, the second coordination encirclement is formed by tiles neighboring tiles from the first encirclement with the exception of tiles from the seed, and so on. Let eq(n) be the nth coordination encirclement. The process of consecutive addition of coordination encirclements we called the layerwise growth process. If there exists the limit $\gamma = \lim_{n\to\infty} \text{eq}(n)/n$, we say that the tiling Til has self-similar growth with the form γ. In this case, γ does not depend on the seed. It should be noted that the problem of correspondence of this model of layerwise growth of the tilings with physical quasicrystal growth is opened and demands additional research.

We prove that the tilings Til(a, b, c, h) have self-similar growth. The growth form is a convex centrosymmetrical hexagon pol(a, b, c) which does not depend on h. Coordinates of its vertices in the hexagonal lattice Γ are

$$\pm\left(\frac{b}{a+b}, \frac{-a}{a+b}\right); \pm\left(1, \frac{a}{a+c}\right); \pm\left(\frac{b}{b+c}, 1\right).$$

The proof consists of two parts. The first part is a bounding growth polygon from below. This part uses parameterization of geodesic chains in the tiling and a method to approximate chains. The details of this technique can be found in the papers of Zhuravlev (2002) and Zhuravlev & Maleev (2007). The second part is a bounding growth polygon from above. It uses a connection between the growth of the tiling Til(a, b, c, h) and the growth of three-dimensional periodic structures, which will be described below.

In Fig. 3, one can see the first 20 coordination encirclements for the tiling Til($1, \sqrt{2}, \sqrt{3}, 0.1$). The seed (black rhombs) is three rhombs of various types nearest to the point of origin.

There exists an interesting connection between the growth of the tiling Til(a, b, c, h) and the growth of three-dimensional periodic structures. Once again, consider the normal tiling of three-dimensional space on unit cubes. Consider the set of faces of all cubes from this tiling. We say that two faces are neighboring if they have a common edge. Similarly to the case of tilings, we can introduce the coordination encirclements. The growth form of this structure is the unit octahedron generated by the vectors $\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3$. Consider the section of this octahedron by the plane $ax + by + cz = 0$. Then the polygon obtained by orthogonal projection of this section to the plane $x + y + z = 0$ is pol(a, b, c).

The strong parameterization of the tiling Til(a, b, c, h) produces an algorithm of layerwise growth for this tiling.

By computer modeling we obtain the following stability property of growth form:

$$\text{eq}(n) \subset (n \cdot \text{pol}(a, b, c))_C.$$

This means that there exists an absolute constant C such that the nth coordination encirclement is a subset of the C-neighborhood of the polygon $n \cdot \text{pol}(a, b, c)$. We also note that the growth hexagon pol(a, b, c) continuously depends on the parameters a, b, c.

Coordination number $m(n)$ of the tiling Til(a, b, c, h) is a cardinality of the set eq(n) or the number of rhombs from the nth coordination encirclement (see Grosse-Kunstleve *et al.*, 1996). Exact topological density (td) of the tiling Til(a, b, c, h)

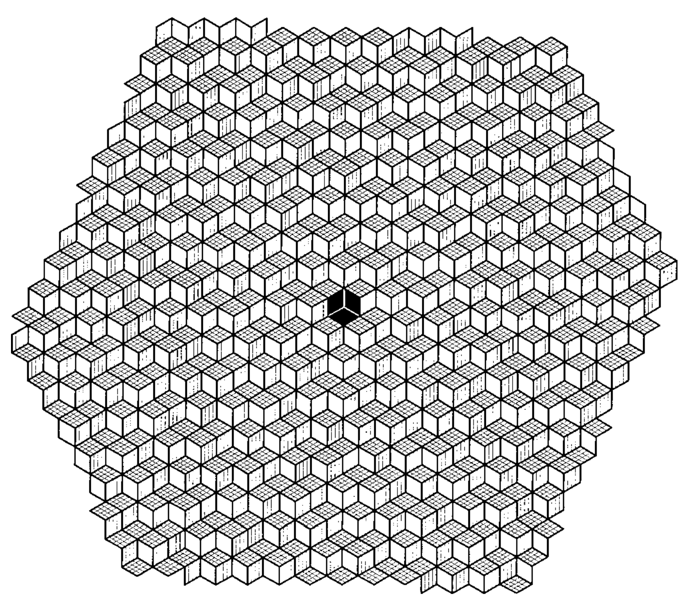

Figure 3
The first 20 coordination encirclements of the seed of three rhombs for the tiling Til($1, \sqrt{2}, \sqrt{3}, 0.1$).

is defined by the equality $\mathrm{td}(a, b, c, h) = \lim_{n\to\infty} m(n)/n$. We prove that

$$\mathrm{td}(a, b, c, h) = 4\left(1 + \frac{abc}{(a+b)(b+c)(c+a)}\right).$$

The proof follows from the formula

$$\mathrm{td}(a, b, c, h) = 2|\mathrm{pol}(a, b, c)|,$$

where $|\mathrm{pol}(a, b, c)|$ is the area of growth polygon. The unit of area is taken as an area of each rhombus. This formula can be proved by standard methods of analytical number theory. Technical details can be found in the paper of Zhuravlev (2002).

From this, we also have an inequality $4 < \mathrm{td}(a, b, c, h) < 4.5$ for any a, b, c. For coordination numbers $m(n)$, we have an asymptotic formula

$$m(n) = \mathrm{td}(a, b, c, h)n + r(n), \quad \text{with } \frac{r(n)}{n} \to 0 \text{ as } n \to \infty.$$

The graph of the function $r(n)$ for the tiling $\mathrm{Til}(1, \sqrt{2}, \sqrt{3}, 0.1)$ is represented in Fig. 4(*a*). This graph was obtained by computer modeling.

To show up some properties of the function $r(n)$, we first consider similar functions for growth sectors.

If we consider rays from the point of origin to vertices of the growth polygon $\mathrm{pol}(a, b, c)$, then we obtain the partition of the plane into six growth sectors. Define a sectorial coordination number $m^{(i)}(n)$ as a number of rhombs in the intersection of the nth coordination encirclement $\mathrm{eq}(n)$ and the ith growth sector. Denote this intersection by $\mathrm{eq}^{(i)}(n)$. Rhombs which intersect with two neighboring sectors are considered to belong to both sectors. Recall that the growth polygon $\mathrm{pol}(a, b, c)$ is centrosymmetrical. Moreover, if the seed is three rhombs of various types nearest to the point of origin, then we can prove that coordination encirclements $\mathrm{eq}^{(i)}(n)$ are centrosymmetrical too. So, sectorial coordination numbers are equal in centrosymmetrical sectors. In the general case, sectorial coordination numbers in centrosymmetrical sectors differ by no more than a constant, depending only on the seed. From this it follows that we can study only the first three sectors.

If we denote by $\mathrm{pol}_i(a, b, c, h)$ the part of the growth polygon located in the ith growth sector, we can prove that $\lim_{n\to\infty} m^{(i)}(n)/n = 2|\mathrm{pol}_i(a, b, c)|$. We will denote this limit as $m^{(i)}$. The proof is similar to the proof of the formula for exact topological density. So we have the following formulas for sectorial coordination numbers:

$$m^{(1,4)}(n) = \frac{a(a+b+c)}{(a+b)(a+c)} n + r^{(1,4)}(n),$$
$$m^{(2,5)}(n) = \frac{c(a+b+c)}{(c+a)(c+b)} n + r^{(2,5)}(n),$$
$$m^{(3,6)}(n) = \frac{b(a+b+c)}{(b+c)(b+a)} n + r^{(3,6)}(n).$$

Here $\lim_{n\to\infty} r^{(i)}(n)/n = 0$. Hence, the function $r^{(i)}(n)$ is the remainder term.

Graphs of the functions $r^{(i)}(n)$ (i = 1, 2, 3), obtained by computer modeling, are represented in Figs. 4(*b*), (*c*), (*d*).

If we want to study the function $r^{(i)}(n)$, we need more information about the sets $\mathrm{eq}^{(i)}(n)$.

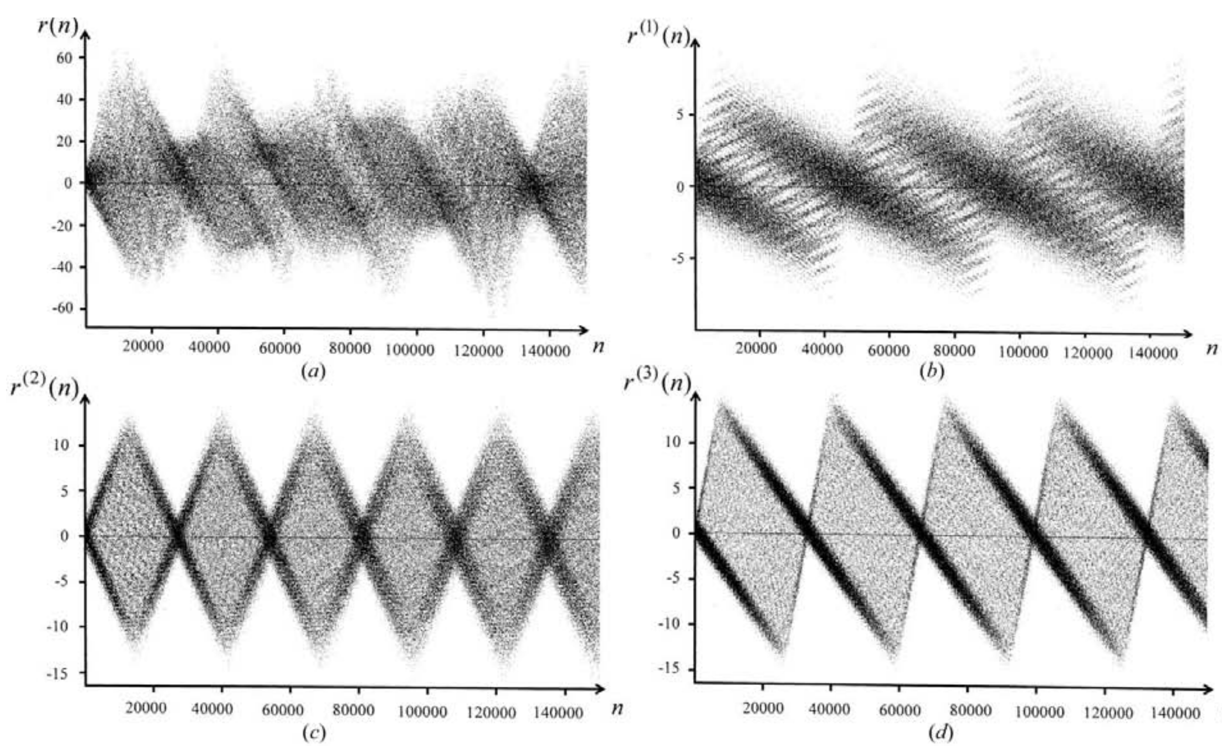

Figure 4
Graphs of the deviation $r(n)$ and sectorial deviations $r^{(i)}(n)$.

Table 2
Parameterization of the sector layers: local numbers.

Sectors I, IV		Sectors II, V		Sectors III, VI	
Half-interval	$\omega^{(1)}$	Half-interval	$\omega^{(2)}$	Half-interval	$\omega^{(3)}$
$(0, a]$	$b - c$	$(0, a]$	b	$(0, a]$	$-a$
$(a, a+b]$	b	$(a, 2a]$	$-a$	$(a, a+b]$	$c - a$
$(a+b, a+b+c]$	$-c$	$(2a, 2a+c]$	$b - a$	$(a+b, a+b+c]$	c
		$(2a+c, a+b+c]$	b		

6. Sector layers

Consider the set $\mathrm{eq}^{(i)}(n)$. Rhombs from $\mathrm{eq}^{(i)}(n)$ are naturally ordered. There exist local numbers with the following property: if x is a parameter of an arbitrary rhomb from $\mathrm{eq}^{(i)}(n)$, then $x \oplus \omega^{(i)}$ is a parameter of the next rhomb. Local numbers are represented in Table 2.

So we have parameterizations of the sets $\mathrm{eq}^{(i)}(n)$. Using the map, we can continue these sets to infinity on one side. Using local numbers, we obtain two-side continuation of the set $\mathrm{eq}^{(i)}(n)$. This continuation is a two-side infinite chain of tiles. This chain is called a sector layer of the ith type. Parameterization of the set $\mathrm{eq}^{(i)}(n)$ continues to the parameterization of a sector layer. In the general case, we can select any rhomb from the tiling $\mathrm{Til}(a, b, c, h)$ with the parameter x and use maps $x \to x \oplus (\omega^{(i)}), x \to x \oplus (-\omega^{(i)})$ on the set of parameters.

We can prove that two sector layers of one type, constructed for one n, coincide. Therefore, the tiling $\mathrm{Til}(a, b, c, h)$ can be decomposed into disjoint sector layers. These sector layers are quasiperiodic. We can regard these sector layers as one-dimensional quasicrystals.

Select any parameter x and consider a patch of sector layer of the ith type, containing the rhomb with parameter x. Suppose that the length of this patch equals l. Denote by $n^{(i)}(x, l)$ the number of rhombs from this patch. We can prove that $\lim_{l\to\infty} n^{(i)}(x, l)/l = m^{(i)}/\tilde{l}^{(i)}$, where $\tilde{l}^{(i)}$ is the length of the ith edge of the growth polygon and $m^{(i)} = \lim_{n\to\infty} m^{(i)}(n)/n$. Then we get the following formulas:

$$n^{(1,4)}(x, l) = \frac{a+b+c}{\sqrt{3a^2 + b^2 + c^2 + 3ab + 3ac + bc}} l + \rho^{(1,4)}(x, l),$$
$$n^{(2,5)}(x, l) = \frac{a+b+c}{\sqrt{a^2 + b^2 + 3c^2 + ab + 3ac + 3bc}} l + \rho^{(2,5)}(x, l),$$
$$n^{(3,6)}(x, l) = \frac{a+b+c}{\sqrt{a^2 + 3b^2 + c^2 + 3ab + ac + 3bc}} l + \rho^{(3,6)}(x, l),$$

where $\lim_{l\to\infty}[\rho^{(i)}(x, l)]/l = 0$.

Consider the remainder terms $\rho^{(i)}(x, l)$. Mathematically, maps $x \to x \oplus \omega^{(i)}$ are an integral transformation of circle rotations $x \to x + \beta^{(i)} (\mathrm{mod}\, 1)$. The definition of integral transformation can be found in the paper of Cornfeld *et al.* (1982). Using computer modeling and mathematical methods from number theory, we can obtain the following properties of the deviations $\rho^{(i)}(x, l)$.

1. *Unbounded growth.* $\rho^{(i)}(x, l) \to \infty$ as $l \to \infty$.

2. *Quasiperiodicity*. There exists an infinite sequence of quasiperiods $P_k^{(i)}$ such that $|\rho^{(i)}(x, l + P_k^{(i)}) - \rho^{(i)}(x, l)| \le 1$ for any l. So the functions $\rho^{(i)}(x, l)$ are close to periodic. However, from the unbounded growth property it follows that the functions $\rho^{(i)}(x, l)$ are non-periodic. Quasiperiods $P_k^{(i)}$ can be computed by the formula $P_k^{(i)} = K^{(i)} Q_k(\beta^{(i)})$. Here $Q_k(\beta^{(i)})$ is the denominator of the kth partial convergent to $\beta^{(i)}$. Numbers $Q_k(\beta^{(i)})$ are easily computed by using a continued fraction expansion of $\beta^{(i)}$. Values of $K^{(i)}$ and $\beta^{(i)}$ are represented in Table 3. Constants can also be computed by the formula $K^{(i)} = [\tilde{l}^{(i)}/m^{(i)}] t^{(i)}$. Here $t^{(i)}$ is an average time of returning the point under the integral transformation into the same interval.

Table 3
Coefficients $K^{(i)}$ and $\beta^{(i)}$ from the formula for quasiperiods of sectorial deviations $\rho^{(i)}(x, l)$.

Sector	$K^{(i)}$	$\beta^{(i)}$
I, IV	$\dfrac{\sqrt{3a^2 + b^2 + c^2 + 3ab + 3ac + bc}}{a+c}$	$\dfrac{a+b}{a+c}$
II, V	$\dfrac{\sqrt{a^2 + b^2 + 3c^2 + ab + 3ac + 3bc}}{b+c}$	$\dfrac{b-a}{b+c}$
III, VI	$\dfrac{\sqrt{a^2 + 3b^2 + c^2 + 3ab + ac + 3bc}}{b+c}$	$\dfrac{c-a}{b+c}$

3. *Growth of amplitude*. Let $A_k^{(i)}$ be the difference between the maximal and minimal values of $\rho^{(i)}(x, l)$ if $1 \le l \le P_k^{(i)}$. Then $A_k^{(i)}$ grows as a sum of partial quotients of the continued fraction expansion of $\beta^{(i)}$. More precisely, we have inequalities

$$C_1^{(i)} \left[k + \sum_{j=1}^{k} q_j(\beta^{(i)}) \right] \le A_k^{(i)} \le C_2^{(i)} \left[k + \sum_{j=1}^{k} q_j(\beta^{(i)}) \right],$$

where $q_j(\beta^{(i)})$ is the jth partial quotient of the continued fraction expansion of $\beta^{(i)}$. Now $C_j^{(i)}$ are constants which do not depend on l and n.

4. *Form of the graph*. If $1 \le l \le P_k^{(i)}$, then we can approximate the graph of the function $\rho^{(i)}(x, l)$ by some polyline $f_k^{(i)}(l)$. This polyline consists of at most three segments. If the polyline consists of only two segments, we will say that this polyline is degenerated. The deviation of the graph from this polyline is bounded by a previous amplitude: $|\rho^{(i)}(x, l) - f_k^{(i)}(l)| \le C_3^{(i)} A_{k-1}^{(i)}$, if $1 \le l \le P_k^{(i)}$.

The proof of 1–4 uses the expansion of remainder terms $\rho^{(i)}(x, l)$ in special sums of fractional parts $\sum_{i=1}^{[l]} \{i\beta^{(i)} + \gamma^{(i,j)}\}$ ($\{x\}$ is a fractional part of x). For this sum, similar properties were proved by Shutov (2006).

From the properties 2–4, we have two corollaries. Firstly, if one of the partial quotients $q_j(\beta^{(i)})$ is relatively big, then we have a very long interval where the graph of the function $\rho^{(i)}(x, l)$ seems to be a periodic polyline. Secondly, the graph of the function $\rho^{(i)}(x, l)$ has fractal structure. In Fig. 5, we represent graphs of the function $\rho^{(3)}(0, l)$ for the tiling $\mathrm{Til}(1, \sqrt{2}, \sqrt{3}, 0.1)$ on various scales. In Table 4 are represented parameters of the continued fraction expansion of $\beta^{(i)}$ and quasiperiods $P_k^{(i)}$.

7. Remainder terms in sectors

Now we apply the obtained results to sectorial remainder terms $r^{(i)}(n)$. Since sets $\mathrm{eq}^{(i)}(n)$ are patches of sector layers, we

Table 4
Partial quotients $q_k^{(i)}$ and denominators of partial convergents $Q_k^{(i)}$ of continued fraction expansion of $\beta^{(i)}$ and quasiperiods $P_k^{(i)}$ of the function $\rho^{(i)}(x, l)$.

k	Sector I			Sector II			Sector III		
	$q_k^{(1)}$	$Q_k^{(1)}$	$P_k^{(1)}$	$q_k^{(2)}$	$Q_k^{(2)}$	$P_k^{(2)}$	$q_k^{(3)}$	$Q_k^{(3)}$	$P_k^{(3)}$
1	1	1	1	7	7	11	4	4	6
2	7	8	13	1	8	13	3	13	20
3	1	9	15	1	15	24	2	30	46
4	1	17	28	2	38	61	1	43	66
5	2	43	70	9	357	578	4	202	310
6	9	404	659	47	16817	27233	95	19233	29522
7	47	19031	31065	1	17174	27811	1	19435	29832
8	1	19435	31724	8	154209	249722	3	77538	119018
9	8	174511	284861	14	2176100	3523921	1	96973	148850

can write the following formula for sector coordination numbers: $m^{(i)}(n) = n^{(i)}(x_n, \tilde{l}^{(i)}(n))$. Here $\tilde{l}^{(i)}(n)$ is the length of a set $\mathrm{eq}^{(i)}(n)$ and x is a parameter of its initial point. Self-similar growth of the tiling $\mathrm{Til}(a, b, c, h)$ implies that

$$\tilde{l}^{(i)}(n) = n\tilde{l}^{(i)} + \tilde{r}^{(i)}(n), \qquad (1)$$

where $\tilde{l}^{(i)}$ is the length of the ith edge of the growth polygon and $|\tilde{r}_n^{(i)}| \leq C_4^{(i)}$. Hence, we have the following relation between the sectorial remainder term $r^{(i)}(n)$ and the layer remainder term $\rho^{(i)}(x, l)$:

$$r^{(i)}(n) = \rho^{(i)}(x_n, \tilde{l}^{(i)}(n)) + \tilde{\rho}^{(i)}(n), \qquad (2)$$

with $|\tilde{\rho}^{(i)}(n)| \leq C_5^{(i)}$.

It can be proved that the initial points x are determined by the equality $x_{n+1} = x_n \oplus \tilde{\omega}^{(i)}$. Local numbers $\tilde{\omega}^{(i)}$ are linearly independent over the field of rational numbers. Hence, points x_n are uniformly distributed on the set of parameters.

Consider the function $\delta^{(i)}(x, n) = \rho^{(i)}(x_n, \tilde{l}^{(i)}(n))$. It can be proved that this function has properties 1–4 as well as the function $\rho^{(i)}(x, l)$ with the renormalization of quasiperiods $\tilde{P}_k^{(i)} = P_k^{(i)}/\tilde{l}^{(i)}$. Functions $\delta_1(n) = \sup_x \tilde{\rho}^{(i)}(x, n)$ and $\delta_2(n) = \inf_x \tilde{\rho}^{(i)}(x, n)$ also have properties 1–4. However, for these functions the approximating polyline is degenerated. Furthermore, $\delta_1(n)$ and $\delta_2(n)$ give us an envelope of the graph $r^{(i)}(n)$. Now we can conclude that the points of the graph $r^{(i)}(n)$ belong to some parallelograms as $1 \leq n \leq \tilde{P}_k^{(i)}$. Further, we must repeat this parallelogram with quasiperiod $\tilde{P}_k^{(i)}$ to the next quasiperiod. Quasiperiod $\tilde{P}_k^{(i)}$ is stable if the partial quotient $q_k^{(i)}$ of continued fraction expansion of $\beta^{(i)}$ is substantially greater then the neighboring partial quotients. For the tiling $\mathrm{Til}(1, \sqrt{2}, \sqrt{3}, 0.1)$, these partial quotients (those underlined in Table 4) determine the stable quasiperiods of an envelope of $r^{(i)}(n)$: $\tilde{P}_7^{(1)} = 45\,944$, $\tilde{P}_6^{(2)} = 26\,526$, $\tilde{P}_6^{(3)} = 32\,832$ (see Figs. 4*b*, *c*, *d*).

Now we consider the question of the distribution of the points of the graph $r^{(i)}(n)$ in the described parallelogram. In the first approximation, x_n is a random point from the set of parameters. The relevance of this assumption is motivated by the uniform distribution of the points x_n. In this case, we can consider the graphs of the functions $\delta^{(i)}(x, n)$ for all x. Then we

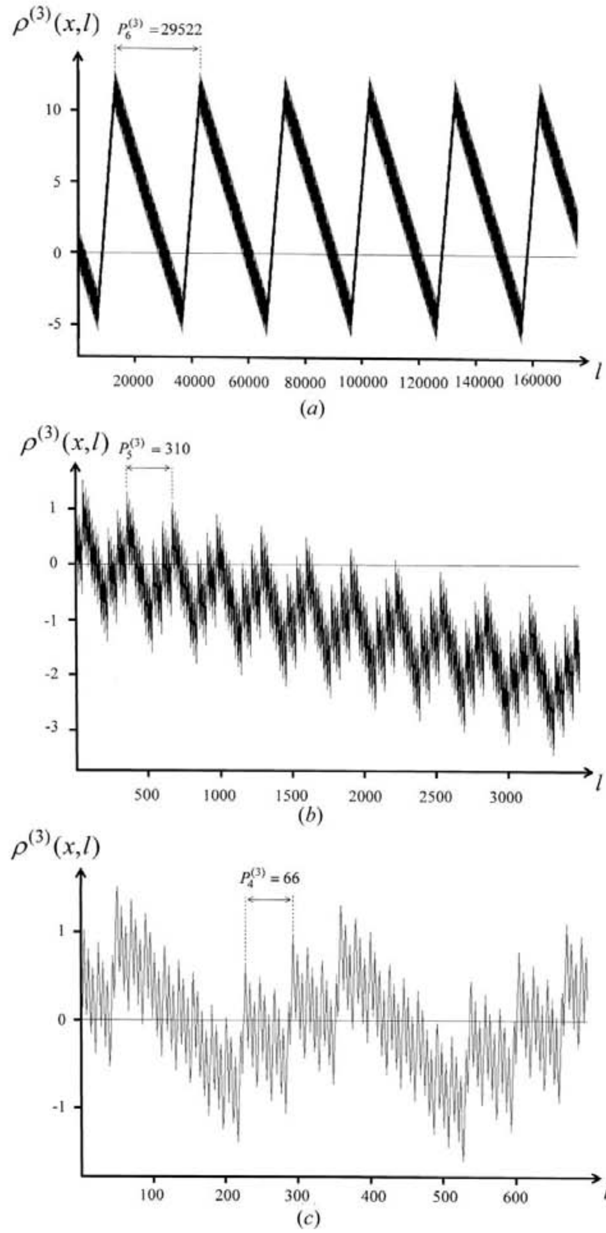

Figure 5
Graphs of the deviation $\rho^{(3)}(0, l)$ on various scales.

can compute the probability measure which describes the distribution of the points on the graph $r^{(i)}(n)$. This measure can be decomposed into two components. The first component is uniformly distributed on parallelograms. The second component has a support in the strip near the boundary of the parallelogram. The width of this strip is equal to $A^{(i)}_{k-1}$. This explains the concentration of the points of the graph $r^{(i)}(n)$ near the parallelogram's boundary and the uniform distribution of such points in the parallelogram (see Figs. 4*b*, *c*, *d*).

8. Conclusions

We have discovered a hexagonal growth form with bounded radius of neighborhood for quasiperiodic Ito–Ohtsuki tilings: $\mathrm{eq}(n) \subset (n \cdot \mathrm{pol}(a, b, c))_C$. This implies a very stable nature of the growth process. For sectorial coordination numbers, we find asymptotic formulas $m^{(i)}(n) = k^{(i)}n + r^{(i)}(n)$ with $\lim_{n\to\infty} r^{(i)}(n)/n = 0$. A graph of the function $r^{(i)}(n)$ has some interesting properties: quasiperiodicity, fractal structure of an envelope, existence of parallelograms, consideration of the points near an envelope and so on. We also discover the existence of one-dimensional sector layers in Ito–Ohtsuki tilings. Properties of these sector layers are similar to the whole tiling. So these sector layers can be considered as one-dimensional quasicrystals.

Note that most of these properties were discovered earlier in the case of the quasiperiodic Rauzy tiling. Ito–Ohtsuki tilings are an infinite family of tilings for which construction is very different from the construction of the Rauzy tiling. So we have several open questions. Are these properties strictly connected with the quasiperiodicity of the tilings? What properties of the tiling, except quasiperiodicity, give us these effects? In particular, are these results true for well known Penrose tilings?

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